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Some Solutions of the Boltzmann Equation Without Angular Cutoff

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We show the existence of local or global in time solutions for the non-homogeneous Boltzmann equation. This is done under the assumptions that initial data are smaller than a suitable Maxwellian and that collisional cross-sections do not satisfy Grad's angular cutoff. Partial regularity in space-velocity of the solutions constructed herein is also proved.

KEY WORDS: Boltzmann; singular cross-sections.

1. INTRODUCTION

In this paper, we consider the Boltzmann equation which consists in looking for a function $f = f(t, x, v)$, $t \in \mathbb{R}^+$ (or in $(0, T)$, with $T > 0$ fixed), $(x, v) \in \mathbb{R}^6$, solution in a suitable sense of

$$\begin{cases} \partial_t f + v.\nabla_x f = Q(f), \\ f|_{t=0} = f_0, \end{cases} \tag{$\mathscr{B}_0$}$$

denoted also hereafter by problem $(\mathscr{B})$.

$f_0 = f_0(x, v)$ is a given initial datum and we assume in this paper that it satisfies

$$m^- M(0, x, v) \leqslant f_0(x, v) \leqslant m^+ M(0, x, v), \tag{1.1}$$

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where $0 < m^- \leqslant m^+$ are given constants and

$$M(t, x, v) = \frac{e^{-|v|^2}}{\sqrt{\pi}^3} \frac{e^{-|x-tv|^2}}{\sqrt{\pi}^3}. \tag{1.2}$$

Let us mention here that the assumption $m^- > 0$ is not necessary for the existence results presented below, but it simplifies the regularity questions dealt with. Very likely, it can be dispensed with in the homogeneous case. But in this paper, we treat the non-homogeneous situation of the Boltzmann equation, that is when f does really depend on the variable x. The Boltzmann operator Q which appears in $(\mathscr{B})$ is given by (for functions $f = f(v)$)

$$Q(f)(v) = \int_{v_* \in \mathbb{R}^3} \int_{S^2_\omega} B\{f'_* f' - f f_*\}, \tag{1.3}$$

where $f' = f(v')$, $f'_* = f(v'_*)$, $f = f(v)$ and $f_* = f(v_*)$. In turn, the so-called post-collisional velocities v' and v'_* are given in function of $(v, v_*) \in \mathbb{R}^6$ and $\omega \in S^2$ as

$$v' = v + (v_* - v, \omega)\, \omega, \qquad v'_* = v_* - (v_* - v, \omega)\, \omega. \tag{1.4}$$

The function B inside the operator Q is called the scattering cross-section and it is of the form $B = B(|v - v_*|, |\frac{v - v_*}{|v - v_*|}.\omega|)$. The above setting is standard, and is explained for instance in [ArBe, CIP, Vil].

Our aim is to show that there exists $T > 0$ so that problem $(\mathscr{B})$ admits weak solutions (to be defined below) in the class $L^1 \cap L^\infty((0, T) \times \mathbb{R}^6_{x,v})$, for initial data satisfying (1.1), and this will hold true for any value of m^+.

The above comparison assumption is classical, and we recall that the function M is a special solution of $(\mathscr{B})$ as $Q(M) = 0$ and $\partial_t M + v.\nabla_x M = 0$.

Such studies already exist in the cutoff case, that is when (roughly speaking) the function $\omega \to B(.,.)$ is in $L^1(S^2)$. When this is the case, one says that Grad's cutoff assumption holds. Classical references are [ArBe, Gou, Ham, IlSh, Lio].

In this work, we deal with the non cut-off case. As far as we know, we are not aware of similar results, except for the papers [Ale1, AlVi], but these ones deal only with so-called renormalised solutions.

Furthermore see for instance [Cer] the non cut-off case is relevant to most physical cases.

In this paper, we will consider two cases of singular cross sections B. The first one is given by

$$\begin{cases} B(|v-v_*|, \cos\theta) = \Phi(|v-v_*|)\, b(\cos\theta), \\ \dfrac{b^-}{|\cos\theta|^\nu} \leqslant b(\cos\theta) \leqslant \dfrac{b^+}{|\cos\theta|^\nu}, \qquad \theta \in (-\pi/2, +\pi/2), \\ b^-, b^+ \text{ constants}, \quad \nu = \dfrac{s+1}{s-1}, \quad 2 < s \leqslant 5, \text{ and with } \gamma = \dfrac{s-5}{s-1}, \\ \Phi(|v-v_*|) = \phi\, |v-v_*|^\gamma \dfrac{1}{1+q\,|v-v_*|^\nu}, \end{cases} \tag{H_1}$$

Above $q > 0$ and $\phi > 0$ are fixed positive constants. Note that $-3 < \gamma \leqslant 0$ and $\frac{3}{2} \leqslant \nu < 3$.

Let us comment on this assumption (H_1). In view of [Cer], the case $q = 0$ corresponds to a pure power-law interaction between particles. For other (more) physical types of interaction, maybe one gets something like the behaviour in (H_1). The fact that we assumed $q > 0$ will be used explicitly in the next section, but we mention that it enables getting uniform bounds. In particular, we get rid of moments of order strictly positive. In some sense, the situation is similar to that of an operator of the type $-|v|^\gamma \Delta_v$ see also the homogeneous framework of the Landau equation [DeVi, Vil].

One can generalise assumption (H_1) in many ways, but the key point is to ask for $\Phi(|v-v_*|)\, |v-v_*|^\nu$ to be bounded for large $|v-v_*|$. One reason for assuming such a behaviour is that we shall look for upper solutions of $(\mathscr{B})$ as $\beta(t)\, M$ (for a suitable function β). In this way, quantities such as $\int_{v_*} \Phi(|v-v_*|)\, |v-v_*|^\nu\, M(v_*)$ enter naturally. Note also that $\gamma + \nu > 0$ as $s > 2$.

The second case of cross section B is given by

$$\begin{cases} B(.,.) = \dfrac{\Theta(|v_* - v'|)}{|v' - v|^\nu}, \qquad \nu = \dfrac{s+1}{s-1}, s > 2, \\ \Theta \in \mathscr{S}(\mathbb{R}), \ = 0 \text{ for small values}, > 0 \text{ otherwise}. \end{cases} \tag{H_2}$$

This kind of assumption has been introduced and explained in [Ale1] for instance. The main advantage is that we can perform some PdO analysis. We refer to Section 3 for more details.

We also introduce for $\varepsilon > 0$, the so-called Fokker–Planck Boltzmann equation which we also study (only) in the case of assumption (H_2). It consists in the following

$$\begin{cases} \partial_t f + v.\nabla_x f - \varepsilon \Delta_v f = Q(f), \\ f|_{t=0} = f_0. \end{cases} \qquad (\mathscr{B}_\varepsilon)$$

Our main motivation for introducing such an academic model is only to study in [Ale2] regularity questions, but of course it is of interest by itself, see for instance [Ham, DiLi2].

Before stating our results, we need first to define our notion of solutions of $(\mathscr{B}_\varepsilon)$.

Definition 1.1. Assume (H_1) or (H_2), and $\varepsilon \geqslant 0$. We say that, for $T > 0$, possibly $T = +\infty$, $f \geqslant 0$ is a weak solution of $(\mathscr{B}_\varepsilon)$ if

$$\begin{cases} f \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^6)), \\ \int_0^b \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_1}} \int_{S^2} B(f'_* f' - f f_*) \ln \frac{f'_* f'}{f f_*} < +\infty, \qquad \forall \text{ finite } b \leqslant T, \\ |v|^2 f \in L^\infty(0, T; L^1(\mathbb{R}^6)), \end{cases}$$

and for all $h \in C_0^\infty([0, T[\times \mathbb{R}^6)$,

$$\int_{[0,T[\times\mathbb{R}^6} f\{-\partial_t h - v.\nabla_x h - \varepsilon \Delta_v h\} = \langle Q(f); h\rangle + \int_{\mathbb{R}^6} f_0 h(0, x, v)\, dx\, dv,$$

where

$$\langle Q(f); h\rangle \equiv \int_{[0,T[\times\mathbb{R}^6} \int_{v_* \in \mathbb{R}^3} \int_{S^2_\omega} B\{f'_* f' - f f_*\}\{h' - h\}$$

$$= \int_{[0,T[\times\mathbb{R}^6} \int_{v_* \in \mathbb{R}^3} \int_{S^2_\omega} B f f_* \{h' - h\}.$$

Remark 1.1. The fact that $\langle Q(f); h\rangle$ for (at least) such h as defined above is meaningful will be explained in the next section but follows also directly from [AlVi, Vil] for instance.

Definition 1.1 applies to both cases (H_1) or (H_2) of collisional cross sections. But when (H_2) holds, one can introduce a (apparently) stronger notion of solution. This fact was already used partially in [Ale1] to define renormalised solutions.

Definition 1.2. Assume (H_2). We say that, for $T>0$, possibly $T=+\infty$, $f \geqslant 0$ is a PdO solution of $(\mathscr{B}_\varepsilon)$ if

$$\begin{cases} f \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^6)), \\ \int_0^b \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}} \int_{S^2} B(f'_* f' - f f_*) \ln \dfrac{f'_* f'}{f f_*} < +\infty, \qquad \forall \text{ finite } b \leqslant T, \\ |v|^2 f \in L^\infty(0, T; L^1(\mathbb{R}^6)), \end{cases}$$

and for all $h \in C_0^\infty([0, T[\times \mathbb{R}^6)$,

$$\int_{[0,T[\times\mathbb{R}^6} f\{-\partial_t h - v.\nabla_x h - \varepsilon \Delta_v h\} = \langle\langle Q(f); h\rangle\rangle + \int_{\mathbb{R}^6} f_0 h(0, x, v)\, dx\, dv,$$

where

$$\langle\langle Q(f); h\rangle\rangle = \langle\langle Q_1(f); h\rangle\rangle + \langle\langle Q_2(f); h\rangle\rangle,$$

$$\langle\langle Q_2(f); h\rangle\rangle = \int_{[0,T[\times\mathbb{R}^6} fh \int_{v_* \in \mathbb{R}^3} \int_{S^2_\omega} B\{f'_* - f_*\}$$

and

$$\langle\langle Q_1(f); h\rangle\rangle = -\int_{[0,T[\times\mathbb{R}^6} f a^*_{t,x}(v, D_v)(h),$$

where we used pdo notations, and $a^*_{t,x}(v, D_v)$ is the adjoint of the operator with symbol $a_{t,x}(v, \xi)$ given by

$$a_{t,x}(v, \xi) = \int_{R^3_\alpha} f(t, x, \alpha + v)\, \tilde{\Theta}(|\alpha|)\, |\alpha \wedge \xi|^{\nu - 1}.$$

where $\tilde{\Theta}$ denotes Θ multiplied by a power of $|\alpha|$.

This concept is used for instance in [Ale1, 4] (but here we do not use the renormalisation process considered therein). In particuliar, we will need the full calculus of PdO from [Mar1, Mar2, Tay1, Tay2].

The main results of the paper are given by

Theorem 1.1. Assume (H_1) and $\varepsilon = 0$. Then, for all $m^+ > 0$, there exists $T \in \mathbb{R}^{+*}$ and two C^1 functions $\delta, \beta: [0, T] \to \mathbb{R}^{+*}$, with $\delta(0) = m^-$ and $\beta(0) = m^+$, such that problem $(\mathscr{B})$ admits a weak solution f satisfying

$$\delta(t)\, M(t, x, v) \leqslant f(t, x, v) \leqslant \beta(t)\, M(t, x, v).$$

Theorem 1.2. Assume (H_2) and $\varepsilon \geqslant 0$. Then

(i) For all $m^+ > 0$, there exists $T \in \mathbb{R}^{+*}$, two C^1 functions δ, β: $[0, T] \to \mathbb{R}^{+*}$, with $\delta(0) = m^-$ and $\beta(0) = m^+$, such that problem $(\mathscr{B}_\varepsilon)$ admits a weak solution f, which is also a PdO solution, satisfying

$$\delta(t)\, M_\varepsilon(t, x, v) \leqslant f(t, x, v) \leqslant \beta(t)\, M_\varepsilon(t, x, v),$$

where

$$M_\varepsilon(t, x, v) = \frac{e^{-\frac{|v|^2}{A}}\, e^{-\frac{A}{D}|x - Bv|^2}}{\sqrt{\pi D}^3},$$

with

$$\begin{cases} A = A(t) = 4\varepsilon t + 1, \\ D_1 = D_1(t) = \dfrac{\varepsilon}{3} t^3 + 1, \\ B = B(t) = \dfrac{\frac{t}{2} + \frac{t}{2} A(t)}{A(t)} \\ D = D(t) = 4\varepsilon t \left(\dfrac{t}{2}\right)^2 + A(t).D_1(t) \end{cases}$$

(ii) If $s > 3$, there exists a constant $C_* > 0$, such that if $0 < m^- \leqslant m^+ \leqslant C_*$, then (i) holds true with $T = +\infty$.

The constant C_* above is denoted by $c_{12,p}$ in Section 3 and is displayed therein.

In fact, a statement similar to Theorem 1.1 holds true also in the case $\varepsilon > 0$ (with assumption (H_1)) as it will be clear from the next sections. However, we only state (and prove) the result for $\varepsilon = 0$.

The last theorem gives a partial regularity result on these solutions.

Theorem 1.3. For $\varepsilon = 0$, the solutions constructed above satisfy

$$hf \in L^2(0, T; H^{\frac{\nu-1}{2\nu}}(\mathbb{R}^6))$$

with $\frac{\nu-1}{2\nu} = \frac{1}{s+1}$, for all $h \in C_0^\infty(]0, T[\times \mathbb{R}^6)$.

The paper is organised as follows. In Sections 2 and 3, we prove the existence Theorems 1.1 and 1.2 respectively. Then, the regularity result is proven in the last Section 4. Let us note that this last result applies to both situations, but in the case when one assumes the second hypothesis (H_2), one can bootstrap this regularity to improve it. As it involves a much more difficult analysis, we refer to [Ale2].

As far as I know, this is the first regularity result for the non-homogeneous Boltzmann equation without cutoff, see also the works of Desvillettes [Des1, 2, DeGo] for the homogeneous case.

We would like to point out that we have considered herein the case where the initial data are bounded by Maxwellians, and our method of proof consists in looking for upper or sub-solutions which are so-called travelling Maxwellians. This explains why we have assumed $q>0$ in hypothesis (H_1). Maybe, looking for upper or sub-solutions with an inverse polynomial behaviour could help for the case $q=0$, see for instance [ArBe] in the cutoff case.

Also, we do not consider herein such questions as unicity, further regularity ... of solutions constructed above. We hope to get back on some of these issues in [Ale2].

As a final remark, it follows from the proofs below that we can reverse the order of presentation in Theorems 1.1 and 1.2.

More precisely, if $T>0$ is fixed, we can construct weak solutions of $(\mathscr{B})$ on the time interval $(0, T)$, if m^+ is sufficiently small. This is a kind of statement in use in the framework of non-linear pde, see for instance [Gou, IlSc] in the case of Boltzmann equation with cutoff. We have chosen the opposite presentation of our results, in the hope of showing the existence of global in time solutions, for any value of m^+. However, even if we have failed in this direction, the solutions constructed herein are renormalised solutions in the sense of [Ale1, AlVi], so that they continue to exist as such for time bigger than T.

2. PROOF OF THEOREM 1.1

For the reader's convenience, we divide it into two steps.

First Step: A cutoff problem

For $n \in \mathbb{N}^*$ fixed, we want to solve the following cutoff problem

$$\begin{cases} \partial_t f + v.\nabla_x f = Q_n(f), \\ f|_{t=0} = f_0, \end{cases} \tag{2.1}$$

where Q_n is the Boltzmann operator corresponding to the cross section

$$B_n = B 1_{|\cos\theta| \geqslant \frac{1}{n}}. \tag{2.2}$$

More precisely, we want to find $T > 0$, two C^1 functions $\delta, \beta: [0, T] \to \mathbb{R}^{+*}$, $\delta(0) = m^-$, $\beta(0) = m^+$, *which do not depend on n*, and such that problem (2.1) admits a weak solution $f(=f^n)$ such that $\delta M \leqslant f \leqslant \beta M$.

We shall use ideas from [Gou, IlSc] in the cutoff case that we adapt to our singular framework.

Note that we could apply (somehow directly) their results in order to solve (2.1), *but* their constants *do depend on n*, and this is definitively useless for our second step, which consists in sending n to $+\infty$.

We shall explicitely display the parameter n on any constant iff it does depend on it.

For any $0 < T \leqslant +\infty$, δ and β given functions in $C^0_+([0, T])$, and l a fixed and given function satisfying

$$\delta(t)\, M \leqslant l \leqslant \beta(t)\, M, \tag{2.3}$$

we introduce the following *linear problem*

$$\begin{cases} \partial_t g + v.\nabla_x g = \iint B_n l' l'_* - \iint B_n l'_* g + g \iint B_n (l'_* - l_*), \\ g|_{t=0} = f_0 \end{cases} \tag{2.4}$$

Note that problem (2.4) writes also

$$\begin{cases} \partial_t g + v.\nabla_x g = \iint B_n l' l'_* - \iint B_n l_* g, \\ g|_{t=0} = f_0 \end{cases} \tag{2.5}$$

Notice that from [Gou] and assumption (H_1), one has for all $t \in [0, T]$

$$\iint B_n l_* \leqslant c_{1,n} \int_{v_*} \Phi(|v - v_*|)\, M(v_*) \leqslant c_{2,n}$$

In fact, one has

$$0 \leqslant \iint B_n l_* \leqslant c_{3,n} \frac{1}{(1+t^2)^{\frac{3+\gamma}{2}}}, \qquad \forall t \in [0, T]$$

In the same way,

$$\iint B_n l' l'_* \leqslant \beta^2(t) \iint B_n M' M'_* \leqslant \beta^2(t) \iint B_n M M_*$$

$$\leqslant \beta^2(t)\, M \iint B_n M_* \leqslant c_{4,n,T}, \qquad \forall T > 0 \text{ fixed.}$$

Finally

$$\int_x \iint B_n l' l'_* \leqslant \int_x \int_v M c_{5,n,T} \leqslant c_{6,n,T}$$

These estimates show that for all $T > 0$ fixed, problem (2.5) admits a unique solution g (in the mild and distributional sense) which is in $L^1 \cap L^\infty((0,T) \times \mathbb{R}^6_{x,v})$ and $\geqslant 0$.

Next, we look for an upper solution of problem (2.5) in the form $a \equiv \beta(t)\, M$. More precisely, we wish to find a sufficient condition on β for this purpose, and so we first compute

$$\partial_t[a-g] + v.\nabla_x[a-g]$$

$$= -\iint B_n l' l'_* + \iint B_n l'_* g - g \iint B_n (l'_* - l_*) + \beta'(t)\, M$$

$$= \beta'(t)\, M - \iint B_n l' l'_* + \iint B_n l'_* g - g \iint B_n (l'_* - l_*)$$

$$= \beta'(t)\, M - \iint B_n l' l'_* + \iint B_n l'_* g + (a-g) \iint B_n (l'_* - l_*) - a \iint B_n (l'_* - l_*).$$

Since $l \leqslant a$, one has

$$\partial_t[a-g] + v.\nabla_x[a-g]$$

$$\geqslant \beta'(t)\, M - \iint B_n l'_* a' + \iint B_n l'_* g + (a-g) \iint B_n (l'_* - l_*) - a \iint B_n (l'_* - l_*)$$

$$\geqslant \beta'(t) M - \iint B_n l'_* (a' - a) - \iint B_n l'_* (a-g)$$

$$+ (a-g) \iint B_n (l'_* - l_*) - a \iint B_n (l'_* - l_*). \tag{2.6}$$

Since we want a to be an upper solution of problem (2.4), it is enough to ask for β to satisfy

$$\beta'(t)\, M - \iint B_n l'_*(a'-a) - a \iint B_n (l'_* - l_*) \geqslant 0. \tag{2.7}$$

In the following, we are led to look for β such that ($\beta \geqslant 0$)

$$\begin{cases} \beta'(t) M \geqslant \iint B_n l'_*(a'-a) + \beta M \iint B_n (l'_* - l_*), \\ \beta(0) \geqslant m^+. \end{cases} \tag{2.8}$$

Firstly, from [ADVW, AlVi, Vil], one has

$$\iint B_n (l'_* - l_*) = \int_{v_*} S_n(|v - v_*|)\, l_* dv_*, \tag{2.9}$$

where

$$S_n(|v-v_*|) = |S^1| \int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{1}{\cos^3\left(\frac{\theta}{2}\right)} \tilde{B}_n \left(\frac{|v-v_*|}{\cos\left(\frac{\theta}{2}\right)}, \cos\theta \right) - \tilde{B}_n(|v-v_*|, \cos\theta)\, d\theta, \right. \tag{2.10}$$

or

$$\begin{aligned} S_n(|v-v_*|) &= |S^1| \int_0^{\frac{\pi}{2}} \sin\theta \frac{1}{\cos^3\left(\frac{\theta}{2}\right)} \left[\tilde{B}_n \left(\frac{|v-v_*|}{\cos\left(\frac{\theta}{2}\right)}, \cos\theta \right) - \tilde{B}_n(|v-v_*|, \cos\theta) \right] d\theta \\ &\quad + |S^1| \int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{1}{\cos^3\left(\frac{\theta}{2}\right)} - 1 \right] \tilde{B}_n(|v-v_*|, \cos\theta)\, d\theta. \end{aligned} \tag{2.11}$$

Above, we have denoted $\tilde{B}_n$ the cross section corresponding to the σ-representation, see for instance [Vil1, 2].

By the results of [ADVW, AlVi, Vil1, 2], one has (using assumption (H_1))

$$|S_n(|v-v_*|)| \leqslant c_1\, |v-v_*|^\gamma \tag{2.12}$$

where c_1 does not depend on n. It follows using [Gou] that

$$\int_{v_*} S_n(|v-v_*|)\, l_* dv_* \leqslant \beta(t)\, c_2 \frac{1}{(1+t)^{3+\gamma}} \tag{2.13}$$

In conclusion, it is enough to choose $\beta \geqslant 0$ such that

$$\begin{cases} \beta'(t)\, M \geqslant \iint B_n l'_*(a'-a) + \beta^2(t)\, M c_2 \dfrac{1}{(1+t)^{3+\gamma}}, \\ \beta(0) \geqslant m^+. \end{cases} \tag{2.14}$$

There remains to analyse the most difficult term $\iint B_n l'_*(a'-a)$. We compute it as follows

$$\begin{aligned} &\iint B_n l'_*(a'-a) \\ &= \phi \iint |v-v_*|^\gamma \frac{1}{1+q\,|v-v_*|^\nu} \frac{1}{\left|\left(\dfrac{v-v_*}{|v-v_*|}, \omega\right)\right|^\nu} 1_{|\cos\theta| \geqslant \frac{1}{n}}\, l'_*(a'-a) \\ &= \phi \iint |v-v_*|^\gamma \frac{|v-v_*|^\nu}{1+q\,|v-v_*|^\nu} \frac{1}{|v'-v|^\nu} 1_{|\cos\theta| \geqslant \frac{1}{n}}\, l'_*(a'-a) \\ &= \phi \iint |v-v_*|^{\gamma+\nu} \frac{1}{1+q\,|v-v_*|^\nu} \frac{1}{|v'-v|^\nu} 1_{|\cos\theta| \geqslant \frac{1}{n}}\, l'_*(a'-a) \\ &= \phi \iint |v'-v_*|^{\gamma+\nu} \frac{1}{1+q\,|v-v_*|^\nu} \frac{1}{|v'-v|^\nu} 1_{|\cos\theta| \geqslant \frac{1}{n}}\, l'_*(a'-a) \\ &\quad + \phi \iint [|v-v_*|^{\gamma+\nu} - |v_* - v'|^{\gamma+\nu}] \frac{1}{1+q\,|v-v_*|^\nu} \frac{1}{|v'-v|^\nu} 1_{|\cos\theta| \geqslant \frac{1}{n}}\, l'_*(a'-a) \\ &= I + II. \end{aligned} \tag{2.15}$$

Note that the bracket term inside II is positive. Furthermore,

$$\begin{aligned} II &\leqslant \phi \iint [|v-v_*|^{\gamma+\nu} - |v_* - v'|^{\gamma+\nu}] \frac{1}{|v'-v|^\nu}\, l'_* a' \\ &\leqslant \beta^2(t)\, M c_3 \int_{v_*} |v-v_*|^\gamma\, M_* \end{aligned}$$

by the computations made in [Ale3]. Finally,

$$II \leqslant \beta^2(t)\, M c_4 \frac{1}{(1+t)^{3+\gamma}}. \tag{2.16}$$

Next, we deal with I. Using the same computations done in [Ale3], and as

$$|v-v_*|^\nu = [|v_* - v'|^2 + |v'-v|^2]^{\frac{\nu}{2}} \geqslant |v_* - v'|^\nu$$

one has successively, using the Carlemann's transform

$$I = \beta(t)\, \phi \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} \int_{E_{0,h}} \frac{|\alpha|^{\gamma+\nu}}{1+q\{|\alpha|^2+|h|^2\}^{\frac{\nu}{2}}} 1\, l'_*(M'-M)$$

where $M' = M(v-h)$ and $M = M(v)$, $E_{0,h}$ denotes the hyperplane orthogonal to h and containing 0. If we let $\bar{M}' = M(v+h)$, $M'_* = M(\alpha+v)$, $M_* = M(\alpha+v-h)$, $\bar{M}_* = M(\alpha+v+h)$, one has

$$\begin{aligned} I &= \beta(t)\, \phi \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} \int_{E_{0,h}} \frac{|\alpha|^{\gamma+\nu}}{1+q\{|\alpha|^2+|h|^2\}^{\frac{\nu}{2}}} 1\, l'_*(M'+\bar{M}'-2M) \\ &\leqslant \beta^2(t)\, c_5 \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} 1_{M'+\bar{M}'\geqslant 2M} \int_{E_{0,h}} \psi(|\alpha|)\, M'_*(M'+\bar{M}'-2M). \end{aligned}$$

Since $M'M'_* = MM_*$ and $\bar{M}'\bar{M}'_* = M\bar{M}_*$, we obtain finally

$$I \leqslant \beta^2(t)\, c_5 M \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} \left| \int_{E_{0,h}} \psi(|\alpha|)\{M_* + \bar{M}_* - 2M'_*\}\, d\alpha \right|, \tag{2.17}$$

where

$$\psi(|\alpha|) = \frac{|\alpha|^{\gamma+\nu}}{1+q\,|\alpha|^\nu}$$

Denote by $\mathscr{C}$ the term inside the $|.|$ in inequality (2.17). Then if $\hat{M}$ denotes the Fourier transform of M with respect to the variable v, one has

$$\mathscr{C} = c_6 \int_{E_{0,h}} d\alpha \psi(|\alpha|) \int_{\mathbb{R}^3} d\xi \hat{M}(\xi)\, e^{i\xi.(\alpha+v)} \sin^2\left(\frac{\xi.h}{2}\right). \tag{2.18}$$

Next, consider for α, $h \neq 0$ fixed

$$\begin{aligned}\mathscr{D} &= \int_{\mathbb{R}^3} d\xi \hat{M}(\xi)\, e^{i\xi.(\alpha+v)} \sin^2\left(\frac{\xi.h}{2}\right)\\ &= \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} dk M(k)\, e^{-ik.\xi} e^{i\xi.(\alpha+v)} \sin^2\left(\frac{\xi.h}{2}\right).\end{aligned} \tag{2.19}$$

We use an orthonormal basis of $\mathbb{R}^3$ with first vector $\frac{h}{|h|}$ and express (2.19) as

$$\begin{aligned}\mathscr{D} = \int_{\mathbb{R}^3} d\xi &\left\{\int_{\mathbb{R}^2_{k\,|\,2,\,3}} M_{2D}(k\,|\,2,3)\, e^{-ik.\xi\,|\,2,\,3}\right\} e^{i\xi.(\alpha+v)\,|\,2,\,3}\\ &\times \int_{\mathbb{R}^1_{k\,|\,1}} M_{1D}(k_1)\, e^{-ik_1\xi_1} e^{i\xi_1 v_1} \sin^2\left(\frac{|h|\,\xi_1}{2}\right),\end{aligned}$$

where$k=(k_i)$,$\xi=(\xi_i)$,$k.\xi|2,3=k_2\xi_2+k_3\xi_3 \ldots M(k)=M_{1D}(k_1)\,M_{2D}(k|2,3)$, M_{1D} or M_{2D} denoting the corresponding Maxwellian in 1 or 2 dimensions. At the end, we obtain

$$\begin{aligned}\mathscr{D} &= M_{2D}((\alpha+v)\,|\,2,3) \int_{\xi_1} \hat{M}_{1D}(\xi_1)\, e^{i\xi_1 v_1} \sin^2\left(\frac{|h|\,\xi_1}{2}\right)\\ &= e^{-|\alpha+S(h)\,v|^2} e^{-|S(h)\,x-t(\alpha+S(h)\,v)|^2} \int_{\xi_1} \hat{M}_{1D}(\xi_1)\, e^{i\xi_1 v_1} \sin^2\left(\frac{|h|\,\xi_1}{2}\right)\end{aligned} \tag{2.20}$$

Above $S(h)$ denotes the orthogonal projection over $E_{0,h}$. Getting back to $\mathscr{C}$ given by (2.18), we obtain

$$\begin{aligned}\mathscr{C} = c_7 &\left\{\int_{E_{0,h}} d\alpha \psi(|\alpha|)\, e^{-|\alpha+S(h)\,v|^2} e^{-|S(h)\,x-t(\alpha+S(h)\,v)|^2}\right\}\\ &\times \int_{\xi_1} \hat{M}^{1D}_{1D}(\xi_1)\, e^{i\xi_1 v_1} \sin^2\left(\frac{|h|\,\xi_1}{2}\right)\end{aligned}$$

and thus

$$|\mathscr{C}| \leqslant c_8 \left|\int_{\xi_1} \hat{M}^{1D}_{1D}(\xi_1)\, e^{i\xi_1 v_1} \sin^2\left(\frac{|h|\,\xi_1}{2}\right)\right| \tag{2.21}$$

in view of the form of ψ and assumption (H_1). Next, we compute crudely as

$$\left|\int_{\xi_1} \hat{M}^{1D}_{1D}(\xi_1)\, e^{i\xi_1 v_1} \sin^2\left(\frac{|h|\,\xi_1}{2}\right)\right| \leqslant c_9 \int_{\xi_1} \frac{1}{(1+t)^{1/2}}\, e^{-\frac{1}{4(1+t^2)}|\xi_1|^2} \sin^2\left(\frac{|h|\,\xi_1}{2}\right). \tag{2.22}$$

If $|h| \leqslant 1$, we bound this as follows

$$\left|\int_{\xi_1} \hat{M}^{1D}_{1D}(\xi_1)\, e^{i\xi_1 v_1} \sin^2\left(\frac{|h|\,\xi_1}{2}\right)\right| \leqslant \frac{c_9}{(1+t^2)^{1/2}} |h|^2 \int_{\xi_1} e^{-\frac{1}{4(1+t^2)}|\xi_1|^2} |\xi_1|^2 \leqslant c_{10}\, |h|^2\, (1+t^2). \tag{2.23}$$

If $|h| \geqslant 1$, we bound $|\sin|$ by 1 to get

$$\left|\int_{\xi_1} \hat{M}^{1D}_{1D}(\xi_1)\, e^{i\xi_1 v_1} \sin^2\left(\frac{|h|\,\xi_1}{2}\right)\right| \leqslant c_{11}. \tag{2.24}$$

Getting back to (2.17), one obtains

$$I \leqslant \beta^2(t)\, M c_{12}(1+t^2). \tag{2.25}$$

Gluing all the above estimates and getting back to (2.8), we are led to choose β such that

$$\begin{cases} \beta'(t) \geqslant \beta^2(t)\, c_{13}(1+t^2), \\ \beta(0) \geqslant m^+ \end{cases} \tag{2.26}$$

One may choose β solution of

$$\begin{cases} \beta'(t) = \beta^2(t)\, c_{13}(1+t^2), \\ \beta(0) = m^+ \end{cases} \tag{2.27}$$

which is given by

$$\beta(t) = \frac{1}{\frac{1}{m^+} - c_{13}(t+t^3/3)}. \tag{2.28}$$

Therefore, if we choose any $T > 0$ such that $\frac{1}{m^+} - c_{13}(T + T^3/3) > 0$, one obtains an upper solution of problem (2.5) for $t \in [0, T]$ in the form βM.

Note that our choice of T and β does not depend on n and this was our main purpose for the computations above. Also we have choosen T such that $\beta(T) < +\infty$.

Once we get at this point, looking for a lower solution in the form $\delta(t)M$ on the same time interval $[0, T]$ is classical, one may also look over to the proof of Theorem 2, given in Section 3. Clearly, we can choose such *δ independent from n.*

In conclusion, we have therefore achieved the following.

There exists $T > 0$ and β, δ C^1 functions from $[0, T]$ in $\mathbb{R}^{+*}$, $\beta(0) = m^+$, $\delta(0) = m^-$, *which do not depend on n*, such that for all

$$l \in L^1 \cap L^\infty((0, T) \times \mathbb{R}^6_{x,v}), \qquad \delta M \leqslant l \leqslant \beta M,$$

problem (2.5) has a unique solution g such that

$$\delta M \leqslant g \leqslant \beta M.$$

By a classical fixed point argument displayed for instance in [Gou], we can assert that there exists g_n solution of the following Boltzmann equation with cutoff

$$\begin{cases} \partial_t g_n + v.\nabla_x g_n = Q_n(g_n), \\ g_n|_{t=0} = f_0, \end{cases} \tag{2.29}$$

on the time interval $[0, T]$, such that $\delta M \leqslant g^n \leqslant \beta M$, where T, δ and β are as above, *not depending on n.*

Furthermore, g_n satisfies the following uniform entropic dissipation rate bound estimate

$$\int_0^T \int_{\mathbb{R}^6} \int_{\mathbb{R}^3_{v*}} B_n\{g'_n g'_{n*} - g_n g_{n*}\} \ln \frac{g'_n g'_{n*}}{g_n g_{n*}} \leqslant C_T, \tag{2.30}$$

as it is clear by multiplying (2.29) by $\ln g^n$.

Second Step: sending n to $+\infty$

From (2.30) and the (uniform in n) L^∞ bound on g^n, one deduces that

$$\int_0^T \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v*}} \int_{S^2_\omega} B_n\{g'_n g'_{n*} - g_n g_{n*}\}^2 \leqslant C_T$$

This is enough to apply the results and the arguments quoted in [ADVW, AlVi]. In particular, there exists $f \in L^1 \cap L^\infty$, $\delta M \leqslant f \leqslant \beta M$ such that (for a suitable sub-sequence)

$$g_n \to f \text{ in } L^p((0,T) \times \mathbb{R}^6_{x,v})$$

strongly ($1 \leqslant p < +\infty$). Writing the distributional formulation associated to (2.29) (as in Definition 1.1), it follows, by the arguments quoted for instance in [AlVi], that f is a weak solution in the sense of Definition 1.1. Note that $Q(f)$ as defined there satisfies

$$Q(f) \in L^2((0,T) \times \mathbb{R}^3_x; H^{-\frac{\nu-1}{2}}(\mathbb{R}^3_v)). \tag{2.31}$$

Indeed for all $h \in L^2((0,T) \times \mathbb{R}^3_x; C_c^\infty(\mathbb{R}^3_v))$

$$\begin{aligned} |\langle Q(f); h\rangle| &= \left| \int_0^T \int_{\mathbb{R}^6} \int_{\mathbb{R}^3_{v_*}} \int_{S^2_\omega} B\{f'f'_* - ff_*\}\{h' - h\} \right| \\ &\leqslant \left\{ \int B\, |f'f'_* - ff_*|^2 \right\}^{1/2} \left\{ \int B\, |h' - h|^2 \right\}^{1/2} \\ &\leqslant C_T\, \|h\|_{L^2((0,T) \times \mathbb{R}^3_x; H^{\frac{\nu-1}{2}}(\mathbb{R}^3_v))}, \end{aligned}$$

as follows from the facts that, on one hand, f is in L^∞ and satisfies the entropic dissipation rate bound in Definition 1.1, and on the other hand by a direct Fourier analysis, using the fact that $B.|v - v_*|^\gamma \leqslant c\, |\cos\theta|^{-\nu}$.

This ends the proof of Theorem 1.

Remark 2.1. Note that (2.31) holds true also in case of assumption (H_2).

3. PROOF OF THEOREM 1.2

As in the proof of Theorem 1.1, and with the same motivation (**non dependence from the parameter** n), the main step consists in solving the following problem, where $\varepsilon \geqslant 0$ and $n \geqslant 1$

$$\begin{cases} \partial_t g + v.\nabla_x g - \varepsilon \Delta_v g = Q_n(g) \\ g|_{t=0} = f_0 \end{cases} \tag{$B_{\varepsilon,n}$}$$

Since we are using some results from [Ham], we assume that

$$m^- M(0,x,v) \leqslant f_0(x,v) \leqslant m^+ M(0,x,v) \tag{3.1}$$

with $0 < m^- \leqslant m^+$ and

$$M(t, x, v) = \frac{e^{-|v|^2}}{\pi^{\frac{3}{2}}} \frac{e^{-|x-tv|^2}}{\pi^{\frac{3}{2}}}. \tag{3.2}$$

The operator Q_n is a Boltzmann cutoff type operator corresponding to the kernel B_n given by

$$B_n = \frac{\Theta(|v_* - v'|)}{|v' - v|^{\nu}} 1_{\frac{|v'-v|}{|v_* - v'|} \geqslant 1/n}. \tag{3.3}$$

Note that from [Ale1,3], B_n does depend on the usual arguments for a cross section.

We shall first consider the case $\varepsilon > 0$.

In the following, we shall display the parameters ε or n on the constants iff they do depend on them.

Denote by $g(t, x, v) \equiv F_\varepsilon(t)\, f_0(x, v)$ the solution of

$$\begin{cases} \partial_t g + v.\nabla_x g - \varepsilon \Delta_v g = 0 \\ g|_{t=0} = f_0 \end{cases} \tag{3.4}$$

Then, it is shown in [Ham] that one has the following

$$0 \leqslant F_\varepsilon(t)\, f_0(x, v) \leqslant m^+ F_\varepsilon(t)\, M(0, x, v). \tag{3.5}$$

If we set

$$M_\varepsilon = M_\varepsilon(t, x, v) = F_\varepsilon(t)\, M(0, x, v), \tag{3.6}$$

then

$$M_\varepsilon(t, x, v) = \frac{e^{-\frac{|v|^2}{A}} e^{-\frac{A}{D}|x - Bv|^2}}{\sqrt{\pi D}^3}, \tag{3.7}$$

with

$$\begin{cases} A = A(t) = 4\varepsilon t + 1, \\ D_1 = D_1(t) = \dfrac{\varepsilon}{3} t^3 + 1, \\ B = B(t) = \dfrac{\frac{t}{2} + \frac{t}{2} A(t)}{A(t)} \\ D = D(t) = 4\varepsilon t \left(\dfrac{t}{2}\right)^2 + A(t).D_1(t) \end{cases} \tag{3.9}$$

Note that $\forall (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^6$, when $\varepsilon \to 0$, then the above quantities go to 1, except for B which goes to t.

Next, let us give $T > 0$, two positive C^1 functions δ and β from $[0, T]$ to $\mathbb{R}^{+*}$, and for all l: $(0, T) \times \mathbb{R}^6 \to \mathbb{R}$, such that

$$\delta(t)\, M_\varepsilon \leqslant l \leqslant \beta(t)\, M_\varepsilon, \tag{3.10}$$

we consider the following linear problem

$$\begin{cases} \partial_t f + v.\nabla_x f - \varepsilon \Delta_v f = \iint B_n (l'_* l' - f l_*) \\ f|_{t=0} = f_0 \end{cases} \tag{3.11}$$

that we also write as

$$\begin{cases} \partial_t f + v.\nabla_x f - \varepsilon \Delta_v f = \iint B_n l'_* (l' - f) + f \iint B_n (l'_* - l_*) \\ f|_{t=0} = f_0 \end{cases} \tag{3.12}$$

Now we are going to specify T, β and δ such that $\beta(t)\, M_\varepsilon$ and $\delta(t)\, M_\varepsilon$ are respectively upper and lower solutions of problem (3.12).

The main point is that we want T, β and δ independent from ε and n.

We begin with the upper one. Since

$$\partial_t M_\varepsilon + v.\nabla_x M_\varepsilon - \varepsilon \Delta_v M_\varepsilon = 0$$

if we set

$$\hat{f} = \beta(t)\, M_\varepsilon, \tag{3.13}$$

we look for $\hat{f}$ such that

$$\partial_t \hat{f} + v.\nabla_x \hat{f} - \varepsilon \Delta_v \hat{f} \geqslant \iint B_n l'_* (l' - \hat{f}) + \hat{f} \iint B_n (l'_* - l_*), \tag{3.14}$$

and thus we are looking for β such that

$$\beta'(t)\, M_\varepsilon \geqslant \beta(t) \iint B_n l'_* (M'_\varepsilon - M_\varepsilon) + \beta(t)\, M_\varepsilon \iint B_n (l'_* - l_*). \tag{3.15}$$

We shall work on each two terms on the right hand side of (3.15) and we first begin with the first one, that is $\iint B_n (l'_* - l_*)$. Using the same computations as in [Ale3], if $E_{0,h}$ denotes the hyperplane through 0 and orthogonal to h, then (using polar coordinates for $h = r\omega$)

$$\iint B_n(l'_* - l_*)$$

$$= \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} \int_{E_{0,h}} 1_{|h| \geq \frac{1}{n}|\alpha|} \times \{2l(\alpha+v) - l(\alpha+v+h) - l(\alpha+v-h)\}\ \Theta(|\alpha|)$$

$$= \int_{\mathbb{R}^3_\alpha} \int_{S^2_\omega} d\omega\ \delta_{\alpha.\omega=0} \int_{\frac{1}{n}|\alpha|}^{+\infty} \frac{1}{r^\nu}\ \Theta(|\alpha|) \times \{2l(\alpha+v) - l(\alpha+v+h) - l(\alpha+v-h)\}$$

$$= \int_{\mathbb{R}^3_\alpha} \int_{S^2_\omega} \Theta(|\alpha|) \int_{\mathbb{R}^3_\xi} \hat{l}(\xi)\ e^{i\xi.(\alpha+v)} \times \int_{\frac{1}{n}|\alpha|}^{+\infty} \frac{2 - e^{-ir\xi.\omega} - e^{ir\xi.\omega}}{r^\nu}\ dr$$

$$= \int_{\mathbb{R}^3_\alpha} \int_{S^2_\omega} \Theta(|\alpha|) \int_{\mathbb{R}^3_\xi} \int_{\mathbb{R}^3_k} l(k)\ e^{-i\xi.k} e^{i\xi.(\alpha+v)} \times \int_{\frac{1}{n}|\alpha|}^{+\infty} \sin^2\left(\frac{r\ |\xi.\omega|}{2}\right) \frac{dr}{r^\nu}$$

$$= \int_{\mathbb{R}^3_k} l(k) \left[\int_{\mathbb{R}^3_\xi} e^{-i\xi.(k-v)} \int_{\mathbb{R}^3_\alpha} \Theta(|\alpha|) \left\{ \int_{S^2_\omega,\ \omega.\alpha=0} \int_{\frac{1}{n}|\alpha|}^{+\infty} \sin^2\left(\frac{r\ |\xi.\omega|}{2}\right) \frac{dr}{r^\nu} \right\} e^{i\xi.\alpha} \right]$$

$$= \int_{\mathbb{R}^3_k} l(k) \left[\int_{\mathbb{R}^3_\xi} e^{-i\xi.(k-v)} I \right]$$

with

$$I = \int_{\mathbb{R}^3_\alpha} \Theta(|\alpha|) \left\{ \int_{S^2_\omega,\ \omega.\alpha=0} \int_{\frac{1}{n}|\alpha|}^{+\infty} \sin^2\left(\frac{r\ |\xi.\omega|}{2}\right) \frac{dr}{r^\nu} \right\} e^{i\xi.\alpha}.$$

Let us analyse I. For $|\xi| \neq 0$, one has (where $S(\alpha)$ denotes the orthogonal projection over $E_{0,\alpha}$)

$$I = \int_{\mathbb{R}^3_\alpha} \Theta(|\alpha|)\ e^{i\xi.\alpha} \int_{S^2_\omega,\ \omega.\alpha=0} |\xi.\omega|^{\nu-1} \int_{\frac{1}{n}|\alpha|\ |\xi.\omega|}^{+\infty} \sin^2\left(\frac{r}{2}\right) \frac{dr}{r^\nu}$$

$$= \int_{\mathbb{R}^3_\alpha} \Theta(|\alpha|) \int_{S^2_\omega,\ \omega.\alpha=0} |S(\alpha)\ \xi.\omega|^{\nu-1} \int_{\frac{1}{n}|\alpha|\ |S(\alpha)\ \xi.\omega|}^{+\infty} \sin^2\left(\frac{r}{2}\right) \frac{dr}{r^\nu}\ e^{i\xi.\alpha}$$

$$= \tilde{\Theta}(|\alpha|) \int_{S^2_\omega,\ \omega.\alpha=0} \||\alpha|\ |S(\alpha)\ \xi.\omega|\|^{\nu-1} \int_{\frac{1}{n}|\alpha|\ |S(\alpha)\ \xi.\omega|}^{+\infty} \sin^2\left(\frac{r}{2}\right) \frac{dr}{r^\nu}\ e^{i\xi.\alpha},$$

where $\tilde{\Theta}$ denotes Θ multiplied by a power of $|\alpha|$.

Let

$$\psi_n(|\alpha|\,|S(\alpha)\,\xi.\omega|) = \psi_n(|\xi|\,|S(\xi)\,\alpha|)$$
$$= \int_{S^2_{\omega},\,\omega.\alpha=0} \||\alpha|\,|S(\alpha)\xi.\omega|\|^{\nu-1} \int_{\frac{1}{n}|\alpha|\,|S(\alpha)\,\xi.\omega|}^{+\infty} \sin^2\left(\frac{r}{2}\right)\frac{dr}{r^{\nu}}.$$

Then, using an orthonormal basis with first vector $\frac{\xi}{|\xi|}$, we get

$$I = \int_{R^3_{\alpha}} \tilde{\Theta}(|\alpha|)\,\psi_n(|\xi|\{\alpha_2^2+\alpha_3^2\}^{\frac{1}{2}})\,e^{i\,|\xi|\,\alpha_1},$$

where $\alpha = (\alpha_i)$. Noticing that

$$|\psi_n(|\xi|\,\{\alpha_2^2+\alpha_3^2\}^{\frac{1}{2}})| \leqslant |\alpha|^{\nu-1}\,|\xi|^{\nu-1},$$

one deduces that I is rapidly decreasing in $|\xi|$. In particular, there exists a constant c_1 (independent of n) such that

$$\left|\iint B_n(l'_*-l_*)\right| \leqslant c_1 \int_{\mathbb{R}^3_k} |l(k)|\,dk, \tag{3.16}$$

and since $l \leqslant \beta M_{\varepsilon}$, one has

$$\left|\iint B_n(l'_*-l_*)\right| \leqslant c_2\beta(t) \int_{\mathbb{R}^3_k} M_{\varepsilon}(k)\,dk. \tag{3.17}$$

In view of this, we need to estimate

$$\mathscr{A} \equiv \int_{\mathbb{R}^3_{\omega}} M_{\varepsilon}(\omega)\,d\omega.$$

By definition, one has the following computations

$$\mathscr{A} = \frac{1}{(\pi D)^{3/2}} \int_{\mathbb{R}^3_{\omega}} e^{\frac{-|\omega|^2}{A}}\,e^{-\frac{A}{D}|x-Bv|^2} = \frac{1}{(\mu D)^{3/2}}\,e^{-\left[\frac{A}{D}-\frac{1}{\mu}\frac{A^2B^2}{D^2}\right]|x|^2}$$

where

$$\mu = \frac{1}{A}+\frac{AB^2}{D}.$$

Note that $\frac{A}{D}-\frac{1}{\mu}\frac{A^2B^2}{D^2} \geqslant 0$ and that $\mu D \geqslant 2t^2$.

After some easy (but long) computations, it follows that

$$\mathscr{A} \leqslant c_3 \frac{1}{(1+t)^3}, \tag{3.18}$$

where c_3 does not depend on n nor on ε by our notation's convention.

In conclusion, we have obtained

$$\beta(t)\, M_\varepsilon \iint B_n(l'_* - l_*) \leqslant \beta^2(t)\, M_\varepsilon c_4 \frac{1}{(1+t)^3}, \tag{3.19}$$

which is the second term on the right hand side of (3.15). There remains to estimate the first one, that is $\beta(t) \iint B_n l'_*(M'_\varepsilon - M_\varepsilon)$.

For notations convenience, we omit the lower index ε below in M_ε (that is we let for a while $M = M_\varepsilon$). Let also $\mathscr{B} = \iint B_n l'_*(M' - M)$. Using Carleman's representation, one has

$$\begin{aligned}
\mathscr{B} &= \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} \int_{E_{0,h}} 1_{|h| \geqslant \frac{1}{n}|\alpha|} \{M(v-h) + M(v+h) - 2M(v)\}\, \Theta(|\alpha|)\, l(\alpha+v) \\
&\leqslant \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} 1_{M(v-h)+M(v+h) \geqslant 2M(v)} \\
&\quad \times \int_{E_{0,h}} \{M(v-h) + M(v+h) - 2M(v)\}\, \Theta(|\alpha|)\, l(\alpha+v).
\end{aligned}$$

Let $M = M(v)$, $M' = M(v-h)$, $\bar{M}' = M(v+h)$, $M'_* = M(\alpha+v)$, $M_* = M(\alpha+v-h)$, $\bar{M}_* = M(\alpha+v+h)$. Then

$$\mathscr{B} \leqslant \beta(t) \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} 1_{M'+\bar{M}' \geqslant 2M} \int_{E_{0,h}} \{M' + \bar{M}' - 2M\}\, M'_* \Theta(|\alpha|).$$

Since $M' M'_* = M M_*$, $\bar{M}' M'_* = M \bar{M}_*$, one gets

$$\begin{aligned}
\mathscr{B} &\leqslant \beta(t) \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} 1_{M'+\bar{M}' \geqslant 2M} \int_{E_{0,h}} \Theta(|\alpha|) \{M_* M + \bar{M}_* M - 2MM'_*\} \\
&\leqslant \beta(t)\, M \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} \left| \int_{E_{0,h}} d\alpha \Theta(|\alpha|) \{M_* + \bar{M}_* - 2M'_*\} \right|
\end{aligned}$$

$$\leqslant \beta(t)\, M \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} \left| \int_{E_{0,h}} d\alpha \Theta(|\alpha|) \int_{\mathbb{R}^3_\xi} \hat{M}(\xi)\, e^{i\xi.\alpha} e^{i\xi.v} \{2 - e^{i\xi.h} - e^{-i\xi.h}\} \right|$$

$$\leqslant \beta(t)\, M \int_{\mathbb{R}^3_h} \frac{dh}{|h|^{\nu+2}} \left| \int_{\mathbb{R}^3_\xi} \hat{M}(\xi) \{2 - e^{i\xi.h} - e^{-i\xi.h}\}\, \hat{\Theta}(|S(h)\xi|) \right|$$

$$\leqslant \beta(t)\, M \int_{S^2_\omega} \int_{\mathbb{R}^3_\xi} |\hat{M}(\xi)|\, |\hat{\Theta}(|S(h)\xi|)|\, |\omega.\xi|^{\nu-1}\, d\xi$$

$$\leqslant \beta(t)\, M \int_{\mathbb{R}^3_\xi} |\hat{M}(\xi)| \left[\int_{S^2_\omega} |\hat{\Theta}(|S(h)\xi|)|\, |\omega.\xi|^{\nu-1} \right]. \tag{3.20}$$

Above, $\hat{\Theta}$ denotes the 2D Fourier transform of Θ. Denote by $\mathscr{C}$ the term in brackets in (3.20). For $|\xi| \neq 0$, one has

$$\mathscr{C} = |\xi|^{\nu-1} \int_{S^2_\omega} |\hat{\Theta}\left(|\xi| \sqrt{1 - \left|\omega.\frac{\xi}{|\xi|}\right|^2} \right) \left|\omega.\frac{\xi}{|\xi|}\right|^{\nu-1}$$

$$\leqslant c_5\, |\xi|^{\nu-1} \int_0^{\frac{\pi}{2}} \sin\theta\, |\hat{\Theta}(|\xi| \sin\theta)|\, (\cos\theta)^{\nu-1}\, d\theta,$$

using the usual polar coordinates. Since Θ is in $\mathscr{S}$, one deduces that for all $0 \leqslant p < 1$, one has

$$|\xi|^{1+p} \int_0^{\frac{\pi}{2}} \sin\theta\, |\hat{\Theta}(|\xi| \sin\theta)|\, (\cos\theta)^{\nu-1}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} |\xi| \sin\theta\, |\hat{\Theta}(|\xi| \sin\theta)|\, |\xi|^p (\sin\theta)^p \frac{(\cos\theta)^{\nu-1}}{(\sin\theta)^p}\, d\theta$$

$$\leqslant c_{6,p}$$

In conclusion

$$\beta(t) \iint B_n l'_*(M'_\varepsilon - M_\varepsilon) \leqslant c_{7,p} \beta(t)\, M_\varepsilon \int_{\mathbb{R}^3_\xi} |\hat{M}_\varepsilon| \frac{|\xi|^{\nu-1}}{(1+|\xi|)^{1+p}}\, d\xi, \tag{3.21}$$

for all $0 \leqslant p < 1$. By computations, one has

$$|\hat{M}_\varepsilon| = \frac{1}{(\mu D)^{3/2}}\, e^{-\left[\frac{A}{D} - \frac{1}{\mu}\frac{A^2 B^2}{D^2}\right]|x|^2} e^{-\frac{1}{4\mu}|\xi|^2}.$$

Note that

$$\frac{|\xi|^{\nu-1}}{(1+|\xi|)^{1+p}} \leqslant \frac{1}{|\xi|^{2-\nu+p}},$$

this being integrable near 0 for $2-\nu+p<3$, and this is the case since $-1<2-\nu<1$ and $0\leqslant p<1$. For the moment, one can simply take $p=0$. Next

$$\begin{aligned}\int_{\mathbb{R}^3_\xi} |\hat{M}_\varepsilon| \frac{|\xi|^{\nu-1}}{(1+|\xi|)^{1+p}}\, d\xi &\leqslant \int_{\mathbb{R}^3_\xi} |\hat{M}_\varepsilon|\, |\xi|^{\nu-2-p}\\ &\leqslant \frac{1}{(\mu D)^{3/2}}\, e^{-\left[\frac{A}{D}-\frac{1}{\mu}\frac{A^2B^2}{D^2}\right]|x|^2} \int_{\mathbb{R}^3_\xi} e^{-\frac{1}{4}\left|\frac{\xi}{\sqrt{\mu}}\right|^2} |\xi|^{\nu-2-p}\\ &\leqslant c_{8,\,p}\, \frac{1}{(1+t)^{2-\nu+p}}\end{aligned}$$

by choosing p near 1 so that $2-\nu+p>0$. Once again, $c_{8,\,p}$ is independent from ε or n.

Note that one can allow any value of $s>2$.

In conclusion, we have obtained

$$\beta(t)\iint B_n l'_*(M'_\varepsilon - M_\varepsilon) \leqslant C_{9,\,p}\beta^2(t)\, M_\varepsilon\left(\frac{1}{1+t}\right)^{2-\nu+p}. \tag{3.22}$$

Getting back to (3.15), we are led to choose $\beta(t)$ (with $\beta(0)\geqslant m^+$) such that

$$\beta'(t)\geqslant C_{9,\,p}\beta^2(t)\left(\frac{1}{1+t}\right)^{2-\nu+p} + c_4\beta^2(t)\left(\frac{1}{1+t}\right)^3, \tag{3.23}$$

where again p is chosen so that $2-\nu+p>0$. Therefore, this reduces to choose β such that (recall that $2-\nu+p<2$)

$$\beta'(t)\geqslant C_{10,\,p}\beta^2(t)\left(\frac{1}{1+t}\right)^{2-\nu+p}. \tag{3.24}$$

At this point, let us first show how to get local solutions for any value of $s>2$.

First, choose $p=0$ and as $-1<2-\nu<1$, we are led to choose β such that (for a suitable constant c_{10})

$$\begin{cases} \beta'(t)\geqslant C_{10}\beta^2(t)\left(\dfrac{1}{1+t}\right)^{2-\nu} \\ \beta(0)\geqslant m^+ \end{cases}. \tag{3.25}$$

It is enough to choose β as the (local) solution of

$$\begin{cases} \beta'(t) = C_{10}\beta^2(t)\left(\dfrac{1}{1+t}\right)^{2-\nu} \\ \beta(0) = m^+ \end{cases}. \tag{3.26}$$

We find, for a suitable constant c_{11} that

$$\beta(t) = \frac{1}{\dfrac{1}{m^+} - c_{11}\{(1+t)^{\nu-1} - 1\}}. \tag{3.27}$$

Note that $0 < \nu - 1 < 2$. This gives us local upper solutions, for all $m^+ \geqslant 0$, up to the time

$$T_{max} = \left\{\frac{1}{m^+ c_{11}} + 1\right\}^{\frac{1}{\nu-1}}$$

and we choose any T, $0 < T < T_{max}$, and β given by (3.27). Note that this choice does not depend on ε nor on n.

To get global upper solutions, we get back to (3.24), with the choice $p \neq 0$, p near 1 so that $2 - \nu + p > 0$. Then, we choose β such that $(\beta(0) = m^+)$

$$\beta(t) = \frac{1}{\dfrac{1}{m^+} - c_{11,\,p} + c_{11,\,p}(1+t)^{\nu-p-1}}.$$

Note that for $s > 3$, one has $\nu - p - 1 < 0$. Therefore, we get global upper solutions for any m^+ such that

$$m^+ \leqslant C_{12,\,p} \equiv \frac{1}{c_{11,\,p}}. \tag{3.28}$$

Next, we look for a lower solution of the form $\hat{g} = \delta(t)\, M_\varepsilon$, knowing that $l \geqslant \delta(t)\, M_\varepsilon$. We are led to look for δ such that

$$\delta'(t) M_\varepsilon \leqslant \delta \iint B_n l'_*(M'_\varepsilon - M_\varepsilon) + \delta M_\varepsilon \iint B_n(l'_* - l_*),$$

and classical arguments show that it is enough to choose δ such that

$$\begin{cases} \delta'(t) \leqslant -c_{10,p}\beta(t)\,\delta(t)(1+t)^{-2+\nu-p}, \\ \delta(t) \leqslant m^- \end{cases} \tag{3.30}$$

in the case $s > 3$ or if $s > 2$ (and $p = 0$)

$$\begin{cases} \delta'(t) \leqslant -c_{10}\beta(t)\,\delta(t)(1+t)^{-2+\nu}, \\ \delta(t) \leqslant m^- \end{cases} \tag{3.31}$$

We can choose δ as solution of (3.30) or (3.31) with equality, and thus we obtain global (resp. local) sub solutions, *which do not depend on* ε *nor on* n.

By classical fixed point arguments, we may arrange for the following and in any case:

for all $0 < m^- \leqslant m^+$, for $T > 0$, $\delta, \beta\colon [0, T] \to \mathbb{R}^{+*}$, with $\delta(0) = m^-$, $\beta(0) = m^+$, constructed above (*independent from* ε *and* n), problem $(\mathscr{B}_{\varepsilon,n})$ admits a weak solution (that is in the sense of definition 1.1 with B replaced by B_n) g_n satisfying

$$\delta M_\varepsilon \leqslant g_n \leqslant \beta M_\varepsilon. \tag{3.32}$$

If $s > 3$, and $m^+ \leqslant c_{12,p}$, one can take $T = +\infty$ as shown above.

Furthermore, one can also manage to get (recall that we assumed $\varepsilon > 0$) $\varepsilon \nabla_v \sqrt{g_n}$ bounded (uniformly with respect to n) in $L^2((0, T) \times \mathbb{R}^6)$ (if $T = +\infty$, locally in time).

To see this, recall that we have obtained g_n as a fixed point of the map which sends $l \in [\delta M_\varepsilon, \beta M_\varepsilon]$ to $f \in [\delta M_\varepsilon, \beta M_\varepsilon]$, f solution of (3.12). Multiplying (using cutoff function in velocity) (3.12) by $\ln f$, one obtains, for *ae* t

$$\int f(t) \ln f(t)\, dx\, dv + \varepsilon \int |\nabla_v \sqrt{f}\,|^2\, dx\, dv = \int B_n(l'_* l' - f l_*) \ln f$$

and this identity is certainly true for any fixed point of the map $l \to f$. Thus replacing l and f above by g_n and using usual manipulations on the collision operator leads to the claim.

We can now end the proof in the case $\varepsilon > 0$:

this is immediate from the arguments of [Lio2], so that we can extract a sub-sequence g_n such that $(1 \leqslant p < +\infty)$

$$g_n \to f \text{ in } L^p((0, T) \times \mathbb{R}^6) \text{ strongly,} \tag{3.33}$$

when $n \to +\infty$, and we obtain that f is a weak solution of $(\mathscr{B}_\varepsilon)$ in the sense of Definition 1.1, by passing to the limit in the weak formulation associated with $(\mathscr{B}_{\varepsilon, n})$.

There remains to prove that f is also a PdO solution in the sense of Definition 1.2. But this follows from the corresponding formulation of $(\mathscr{B}_{\varepsilon, n})$, see for instance [Ale1, 4].

There remains to show the case $\varepsilon = 0$.

There are (at least) two ways to perform this case. One consists in repeating the above process with $\varepsilon = 0$. In fact most of the above computations are true in this case, except for the final argument we used to pass to the limit with n. The second way is a little more painless, and consists in passing to the limit when $\varepsilon \to 0$ with the solutions constructed above. In any case, one needs an extra argument of compactness. We will follow the second way and only detail the case of local solutions.

Firstly, recall from above that for all $0 < m^- \leqslant m^+$, for all $\varepsilon > 0$, we have constructed $0 < T < +\infty$, $\delta, \beta : [0, T] \to \mathbb{R}^{+*}$ C^1 functions, and this independently from ε, such that problem $(\mathscr{B}_\varepsilon)$ admits a weak solution f^ε in the sense of definition 1.1, and such that $\delta M_\varepsilon \leqslant f^\varepsilon \leqslant \beta M_\varepsilon$.

We will show the strong compactness of f^ε in any $L^p((0, T) \times \mathbb{R}^6)$ and this will be enough to conclude the proof in the case $\varepsilon = 0$, see [ADVW, AlVi, Lio2, Lio3].

Some parts below are extracted from our papers.

Obviously, f^ε is bounded (uniformly wrt ε) in $L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^6))$ and any weak limit value will satisfy $\delta M \leqslant f \leqslant \beta M$.

Next, by Definition 1.1, since f^ε has a dissipation rate bounded uniformly wrt ε, it follows (C is any constant independent from ε) using the notation $g = f^\varepsilon$

$$\int_0^T \int_x \int_v \int_{v*} \int_\omega B\,|g' g'_* - g g_*|^2 \leqslant C_T. \tag{3.34}$$

Next, we write first

$$|g' g'_* - g g_*|^2 = (g_* g' - g g'_*)^2 + (g'^2 - g^2)(g'^2_* - g^2_*). \tag{3.35}$$

and we consider the estimate (3.34) involving the second term in (3.35). We claim that it is bounded. By the usual change of variables, this is equivalent to show that

$$\left| \int_0^T \int_x \int_v \int_{v*} \int_\omega B g^2 (g^{2\prime}_* - g^2_*) \right| \leqslant C. \tag{3.36}$$

Indeed, letting $\mathscr{B}$ for the l.h.s of (3.36), one has first

$$\mathscr{B} \leqslant \int_{t,x} (sup_v\, g^2) \int_v \left| \int_{v*} \int_\omega B(g_*^{2\prime} - g_*^2) \right|, \tag{3.37}$$

and as $\int_{v*} \int_\omega B(g_*^{2\prime} - g_*^2)$ is of the same form as considered earlier (see (3.17) for l), it follows $\mathscr{B} \leqslant C\, \|g^2\|_{L^1_{t,x,v}}$. In view of this estimate, of (3.34) and (3.35), one deduces

$$\int_0^T \int_x \int_v \int_{v*} \int_\omega B\, |g_* g' - g g_*'|^2 \leqslant C_T. \tag{3.38}$$

Next, we use the Carleman's representation to get from this

$$\int_0^T \int_x \int_v \int_h \frac{dh}{|h|^{\nu+2}} \int_{E_{0,h}} \bar{\Theta}(|\alpha|) \{g(\alpha+v-h)\, g(v-h) - g(\alpha+v)\, g(v)\}^2 \leqslant C, \tag{3.39}$$

where $\bar{\Theta}$ denotes Θ multiplied by a power of $|\alpha|$. Setting

$$j(z, \alpha) = g(\alpha+z)\, g(z), \tag{3.40}$$

and using the Parseval's relation with respect to the variable v, one gets

$$\int_0^T \int_x \int_h \frac{dh}{|h|^{\nu+2}} \int_{E_{0,h}} \bar{\Theta}(|\alpha|) \int_k |\hat{j}^1(k, \alpha)|^2\, |e^{-ih.k} - 1|^2 \leqslant C, \tag{3.41}$$

($\hat{j}^1$ denotes the F-transform w.r.t to the variable z) that is also

$$\int_0^T \int_x \int_{S^2_\omega} \int_{E_{0,\omega}} \bar{\Theta}(|\alpha|) \int_k |\hat{j}^1(k, \alpha)|^2\, |k.\omega|^{\nu-1} \leqslant C, \tag{3.42}$$

or, using previous notations

$$\int_0^T \int_x \int_\alpha \bar{\Theta}(|\alpha|) \int_k |\hat{j}^1(k, \alpha)|^2\, |S(\alpha).k|^{\nu-1} \leqslant C. \tag{3.43}$$

We claim that

$$\int_0^T \int_x \int_k \left[\int_\alpha \bar{\Theta}(|\alpha|)\, |\hat{j}^1(k, \alpha)|\, |k|^{\frac{\nu-1}{2}} \right]^2 \leqslant C. \tag{3.44}$$

Indeed, letting $\mathscr{A}$ for the left hand side of (3.44), one has

$$\mathscr{A} = \int_0^T \int_x \int_k \left[\int_\alpha \bar{\Theta}(|\alpha|)\, |\hat{j}^1(k,\alpha)|\, |S(\alpha).k|^{\frac{\nu-1}{2}} . \frac{|k|^{\frac{\nu-1}{2}}}{|S(\alpha).k|^{\frac{\nu-1}{2}}} \right]^2,$$

which, using Cauchy–Schwarz inequality with respect to the variable α gives

$$\mathscr{A} \leqslant \int_0^T \int_x \int_k \left\{ \int_\alpha \bar{\Theta}(|\alpha|)\, |\hat{j}^1(k,\alpha)|^2\, |S(\alpha).k|^{\nu-1} \right\} \times \left\{ \int_\alpha \bar{\Theta}(|\alpha|)\, \frac{|k|^{\nu-1}}{|S(\alpha).k|^{\nu-1}} \right\}. \tag{3.45}$$

But

$$\int_\alpha \bar{\Theta}(|\alpha|)\, \frac{|k|^{\nu-1}}{|S(\alpha).k|^{\nu-1}} = \int_\alpha \bar{\Theta}(|\alpha|)\, \frac{1}{|S(k).\alpha|^{\nu-1}} \leqslant C,$$

by assumptions on Θ (note also that $0 < \nu - 1 < 2$). Therefore, it follows that

$$\mathscr{A} \leqslant C \int_0^T \int_x \int_k \int_\alpha \bar{\Theta}(|\alpha|)\, |\hat{j}^1(k,\alpha)|^2\, |S(\alpha).k|^{\nu-1}, \tag{3.46}$$

and the right hand side of (3.46) is bounded in view of (3.43), leading to (3.44). From this, it follows

$$\int_0^T \int_x \int_k |k|^{\nu-1} \left| \int_\alpha \bar{\Theta}(|\alpha|)\, \hat{j}^1(k,\alpha) \right|^2 \leqslant C. \tag{3.47}$$

Note that $|\hat{j}^1(k,\alpha)|$ is up to dilatation in α the modulus of the Wigner transform of g (and thus bounded in $L^2_{k,\alpha} \cap L^\infty_{k,\alpha}$).

Finally, we have obtained that $(f^\varepsilon *_v \tilde{\Theta})\, f^\varepsilon$ belongs to a bounded (wrt ε) set of $L^2((0,T) \times \mathbb{R}^3_x; H^{\frac{\nu-1}{2}}(\mathbb{R}^3_v))$.

At this point, again the arguments of Lions [Lio1, 2, 3] apply to the sequence f^ε, and there exists f, $\delta M \leqslant f \leqslant \beta M$, such that, up to a subsequence ($1 \leqslant p < +\infty$)

$$f^\varepsilon \to f \text{ strongly in } L^p((0,T) \times \mathbb{R}^6)$$

Remark 3.1. One can also use the lower bound on f^ε to deduce compactness wrt variable v, as in [Vil2], instead of the above argument.

4. PROOF OF THEOREM 1.3

Since f is a weak solution in the sense of Definition 1.1, more precisely as it satisfies the entropic dissipation rate bound, and as it is bounded below and above by a Maxwellian, it follows from [ADVW, Vil1, 2], that one has for all $h \in C_c^\infty((0,T) \times \mathbb{R}^6_{x,v})$

$$hf \in L^2((0,T) \times \mathbb{R}^3_x; H^{\frac{\nu-1}{2}}(\mathbb{R}^3_v)). \tag{4.1}$$

In the following, set $F = hf$. Then it satisfies

$$\partial_t F + v.\nabla_x F = hQ(f) + f[\partial_t h + v.\nabla_x h].$$

By the entropy inequality, see Section 2 or 3, it follows also that

$$hQ(f) \in L^2((0,T) \times \mathbb{R}^3_x; H^{\frac{-(\nu-1)}{2}}(\mathbb{R}^3_v)). \tag{4.2}$$

Let $p = \nu - 1$. If $\hat{}$ denotes the Fourier transform with respect to the variables (x, v) and (ξ, μ) the dual variables, then letting $G = hQ(f)$, $H = f[\partial_t h + v.\nabla_x h]$, one has

$$\partial_t \hat{F} - \xi.\nabla_\mu \hat{F} = \hat{G} + \hat{H}. \tag{4.3}$$

By (4.2) and (4.1), we can write

$$\hat{G} + \hat{H} = \hat{g}_1 + |\mu|^{p/2}\, \hat{g}_2, \tag{4.4}$$

where g_i belong to L^2. On each side of (4.3),we add $|\mu|^p\, \hat{F}$ to get

$$\partial_t \hat{F} - \xi.\nabla_\mu \hat{F} + |\mu|^p\, \hat{F} = \hat{g}_1 + |\mu|^{p/2}\, \hat{g}_2 + |\mu|^p\, \hat{F}. \tag{4.5}$$

By (4.1) $|\mu|^{p/2}\, \hat{F} \in L^2$. Therefore, one may write

$$\partial_t \hat{F} - \xi.\nabla_\mu \hat{F} + |\mu|^p\, \hat{F} = \hat{g}_3 + |\mu|^{p/2}\, \hat{g}_4, \tag{4.6}$$

where $g_i \in L^2$.

At this point, one can proceed as in [Per] to get

$$\partial_t\, |\hat{F}|^2 - \xi.\nabla_\mu\, |\hat{F}|^2 + |\mu|^p\, |\hat{F}|^2 \leqslant |\hat{F}\hat{g}_3| + |\mu|^p\, |\hat{F}|^2 + |\hat{g}_4|^2, \tag{4.7}$$

and thus

$$|\hat{F}(t, \xi, \mu)|^2 \leqslant |\hat{F}_0(\xi, \mu + t\xi)|^2 + \int_0^t (|\hat{F}\hat{g}_3| + |\hat{g}_4|^2)(\xi, \mu + s\xi, t - s)\, ds. \tag{4.8}$$

Fix $r \geqslant 0$ and $D \geqslant 0$. Then

$$\begin{aligned}
&\int_0^T dt \int_{\mathbb{R}^3_\mu} |\xi|^r \, |\hat{F}(t,\xi,\mu)|^2 \\
&\quad \leqslant \int_0^T dt \int_{|\mu|\geqslant D} |\xi|^r \, |\hat{F}(t,\xi,\mu)|^2 + \int_0^T dt \int_{|\mu|\leqslant D} |\xi|^r \, |\hat{F}(t,\xi,\mu)|^2 \\
&\quad \leqslant \int_0^T dt \int_{|\mu|\geqslant D} |\xi|^r \, |\hat{F}(t,\xi,\mu)|^2 + \int_0^T dt \int_{|\mu|\leqslant D} |\xi|^r \, |\hat{F}_0(\xi,\mu+t\xi)|^2 \\
&\qquad + \int_0^T dt \int_{|\mu|\leqslant D} \int_0^t (|\hat{F}\hat{g}_3| + |\hat{g}_4|^2)(\xi, \mu+s\xi, t-s)\, ds \\
&\quad \equiv I + II + III. \qquad (4.9)
\end{aligned}$$

One has

$$\begin{aligned}
II &= \int_0^T dt \int_{|\mu|\leqslant D} |\xi|^r \, |\hat{F}_0(\xi,\mu+t\xi)|^2 \\
&= \int_0^T dt \int_{|\mu-t\xi|\leqslant D} |\xi|^r \, |\hat{F}_0(\xi,\mu)|^2 \\
&\leqslant \int_0^T dt \int_{\mathbb{R}^3_\mu} 1_{\left|t-\frac{|\mu|}{|\xi|}\right| \leqslant \frac{D}{|\xi|}} |\hat{F}_0(\xi,\mu)|^2 \\
&\leqslant |\xi|^{r-1} D \int_{\mathbb{R}^3_\mu} |\hat{F}_0(\xi,\mu)|^2. \qquad (4.10)
\end{aligned}$$

In the same way, one gets

$$III \leqslant |\xi|^{r-1} D \int_{\mathbb{R}^3_\mu \times (0,T)} (|\hat{F}\hat{g}_3| + |\hat{g}_4|^2)(s,\xi,\mu)\, ds\, d\mu. \qquad (4.11)$$

In conclusion, one obtains

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3_\mu} |\xi|^r \, |\hat{F}(t,\xi,\mu)|^2 \, dt\, d\mu \\
&\qquad \leqslant |\xi|^{r-1} D\mathscr{A} + \int_0^T \int_{|\mu|\geqslant D} |\hat{F}|^2 \, d\mu\, dt, \qquad (4.12)
\end{aligned}$$

where

$$\mathscr{A}=\int_{\mathbb{R}^3_\mu}|\hat{F}_0(\xi,\mu)|^2+\int_{\mathbb{R}^3_\mu\times(0,T)}(|\hat{F}\hat{g}_3|+|\hat{g}_4|^2)(s,\xi,\mu)\,ds\,d\mu.$$

Next, since

$$\int_0^T\int_{\mathbb{R}^3_\mu}|\xi|^r\,|\hat{F}(t,\xi,\mu)|^2\,dt\,d\mu$$

$$\leqslant|\xi|^{r-1}\,D\mathscr{A}+\frac{|\xi|^r}{D^p}\int_0^T\int_{\mathbb{R}^3_\mu}|\mu|^p\,|\hat{F}|^2\,dt\,d\mu, \tag{4.13}$$

by arguments from [Per], choosing $D=|\xi|^{\frac{1}{1+p}}$ and $r=\frac{p}{1+p}$, we get

$$\int_0^T\int_{\mathbb{R}^3_\mu}|\xi|^r\,|\hat{F}(t,\xi,\mu)|^2\,dt\,d\mu\leqslant C. \tag{4.14}$$

Noticing that $r=\frac{\nu-1}{\nu}=\frac{2}{s+1}$, we have obtained finally that

$$hf\in L^2(0,T;H^{\frac{1}{s+1}}(\mathbb{R}^3_x\times\mathbb{R}^3_v)), \tag{4.15}$$

and this concludes the proof of the regularity result.

Remark 4.1. It is clear that any improvement of the regularity will have to deal with a detailed functional analysis of $Q(f)$. This will be done in [Ale2].

REFERENCES

[Ale1] R. Alexandre, Une définition des solutions renormalisées pour l'équation de Boltzmann, *Note C.R.A.S Paris, t. 328, Série I,* pp. 987–991 (1999).

[Ale2] R. Alexandre, work in preparation.

[Ale3] R. Alexandre, *Around 3D Boltzmann operator without cutoff. A New formulation. M^2AN*, Vol. 34, No. 3, pp. 575–590 (2000).

[Ale4] R. Alexandre, From Boltzmann to Landau, *SIAM J. Appl. Maths.* (1999), submitted.

[ADVW] R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg, Entropy dissipation and long range interactions, *Arch. Rat. Mech. Anal.* **152**(4):327–355 (2000).

[AlVi] R. Alexandre and C. Villani, On the Boltzmann equation with long range interactions and the Landau approximation in plasma physics. Preprint DMI-ENS Paris (1999). First part submitted to C.P.A.M.

[ArBe] L. Arlotti and N. Bellomo, *Lectures Notes on the Math. Theory of the Boltzmann Equation*, N. Bellomo, ed., Vol. 33 (World Sc., 1995).

[Bal] R. Balescu, *Statistical Mechanics of Charged Particles* (Wiley Interscience, N.Y., 1963).

[Cer] C. Cercignani, *Mathematical Methods in Kinetic Theory*, 2nd ed. (Plenum, 1990).

[CIP] C. Cercignani, R. Illner, and M. Pulvirenti, The Mathematical Theory of Dilute Gases, series in *Appl. Sc.*, Vol. 106 (Springer Verlag, New York, 1994).

[Des1] L. Desvillettes, Regularisation for the non cutoff 2D radially symmetric Boltzmann equation with a velocity dependent cross section, *Transp. Theory and Stat. Phys.* **25**(3–5):383–394 (1996).

[Des2] L. Desvillettes, Regularisation properties of the 2D non radially symmetric non cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules, *Transp. Theory and Stat. Phys.* **26**(3):341–357 (1997).

[DeGo] L. Desvillettes and F. Golse, On a model Boltzmann equation without angular cutoff, preprint, Univ. Orléans 98-01 (1998).

[DeVi] L. Desvillettes and C. Villani, On the spatially homogeneous Landau equation for hard potentials, preprint DMI-ENS Paris (1998).

[DiLi1] R. J. DiPerna and P. L. Lions, On the Cauchy problem for Boltzmann equation; Global existence and weak stability, *Ann. Maths.* **130**:321–366 (1989).

[DiLi2] R. J. DiPerna and P. L. Lions, On the Fokker–Planck Boltzmann equation, *Com. Math. Phys.* **120**:1–23 (1988).

[Gou] T. Goudon, Generalized invariants sets for the Boltzmann equation, M^3AS **7**(4):457–476 (1997).

[Ham] K. Hamdache, Sur l'existence globale de solutions de l'équation de Boltzmann. Thèse d'Etat. Univ. Paris 6, Paris (1986).

[IlSh] R. Illner and M. Shinbrot, The Boltzmann equation. Global existence for a rare gas in an infinite vacuum, *Com. Math. Phys.* **95**:217–226 (1984).

[Lio1] P. L. Lions, Compactness in Boltzmann's equation, via FIO and applications, *J. Math. Kyoto Univ.* Part I **34**:391–427; Part II **34**:429–461; Part III **34**:539–584 (1994).

[Lio2] P. L. Lions, On Boltzmann and Landau equation, *Phil. Trans. Roy. Soc. London A* **346**:191–204 (1994).

[Lio3] P. L. Lions, Régularité et compacité pour des noyaux de collisions de Boltzmann sans troncature angulaire, *Note C.R.A.S Paris, t. 326, Série I*, pp. 37–41 (1998).

[Mar1] J. Marschall, Pdo with non regular symbols of the class $S^m_{\varrho,\delta}$, *Com. Part. Diff. Equ.* **12**(8): 921–965 (1987).

[Mar2] J. Marschall, Pdo with coefficients in Sobolev spaces, *Trans. AMS* **307**(1):335–361 (1988).

[Per] B. Perthame, Higher moments for kinetic equations for Vlasov–Poisson and Fokker–Planck cases, *M2AS* **13**:441–452 (1990).

[Tay1] M. E. Taylor, *Pseudo-Differential Operators* (Princeton University Press, 1981).

[Tay2] M. E. Taylor, *Pdo and Non Linear PDE* (Birkhauser, Boston, 1991).

[Tay3] M. E. Taylor, *Partial Differential Equations*, Vols. I, II, and III. *Appl. Math. Sc.*, Vols. 115, 116, and 117 (Springer Verlag, 1996).

[Tri1] H. Triebel, *Theory of Functions Spaces*, Vol. I. Monog. in Maths., Vol. 78 (Birkhauser, 1983).

[Tri2] H. Triebel, *Theory of Functions Spaces*, Vol. II. Monog. in Maths., Vol. 84 (Birkhauser, 1992).

[Tri3] H. Triebel, *Interpolation Theory, Functions Spaces, Differential Operators* (North-Holland, Amsterdam, 1978).

[Vil1] C. Villani, Thèse Université Paris-Dauphine, Paris (1998).

[Vil2] C. Villani, Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equation without cut-off, *Rev. Mat. Iberoam*, to appear.